

## TROPICAL DECOMPOSITION OF YOUNG'S PARTITION LATTICE

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ABSTRACT. Young's partition lattice  $L(m, n)$  consists of unordered partitions having  $m$  parts where each part is at most  $n$ . Using methods from complex algebraic geometry, R. Stanley proved that  $L(m, n)$  is rank-symmetric, unimodal, and strongly Sperner. Moreover, he conjectured that  $L(m, n)$  has a stronger property called symmetric chain decomposition. Despite many efforts, this conjecture has only been proved for  $\min(m, n) \leq 4$ . In this paper, we use tropical polynomials derived from the secant varieties of the rational normal curve in  $\mathbb{P}^n$  in order to construct a canonical decomposition of  $L(m, n)$  into centered subposets  $Q(m, n, e_1, \dots, e_k)$  which are symmetric chain orders if  $m$  is sufficiently large. In particular, we obtain a symmetric chain decomposition for the subposet of  $L(m, n)$  consisting of "sufficiently generic" elements.

## 1. INTRODUCTION

Young's partition lattice  $L(m, n)$  is defined to be the poset of unordered partitions  $\lambda = (0 \leq \lambda_1 \leq \dots \leq \lambda_m \leq n)$  equipped with the following partial order:

$$\lambda \leq \mu \iff \lambda_i \leq \mu_i \text{ for } 1 \leq i \leq m.$$

This poset appears in different guises in several branches of mathematics. For example,  $L(m, n)$  is isomorphic to the poset of Schubert cells in the Grassmannian of  $m$ -planes in  $\mathbb{C}^{m+n}$ . In the groundbreaking paper [4], R. Stanley applied the hard Lefschetz theorem to prove that  $L(m, n)$  is rank-symmetric, unimodal, and strongly Sperner. Furthermore, he conjectured that  $L(m, n)$  has a *symmetric chain decomposition*, i.e. can be expressed as a disjoint union of rank-symmetric saturated chains. Even after many years, this conjecture has only been proved for  $\min(m, n) \leq 4$  [2, 7].

It turns out that  $L(m, n)$  is isomorphic to the poset  $A_n(m)$  of monomials of degree  $m$  in  $A_n = \mathbb{C}[z_0, \dots, z_n]$ , where the partial order is defined by the usual  $\mathfrak{sl}_2\mathbb{C}$ -action. The advantage of working with this "symmetric" model is that we can directly apply tools from representation theory and commutative algebra. Indeed, the initial ideals (with respect to any diagonal term order) of the secant ideals of the rational normal curve in  $\mathbb{P}^n$  give us a canonical set of squarefree monomial ideals  $I_{n,r} \subset A_n$ , for  $1 \leq r \leq k = \lfloor \frac{n}{2} \rfloor$ . We construct tropical polynomials  $f_{n,r} : \mathbb{Z}_{\geq 0}^{n+1} \rightarrow \mathbb{Z}_{\geq 0}$  such that

$$z_0^{a_0} \dots z_n^{a_n} \in I_{n,r}^{(s)} \iff f_{n,r}(a_0, \dots, a_n) \geq s$$

where  $I_{n,r}^{(s)}$  denotes the  $s$ -th symbolic power of  $I_{n,r}$ . We define the tropical decomposition of  $L(m, n)$  as follows:

$$Q(m, n, e_1, \dots, e_k) = \{z_0^{a_0} \dots z_n^{a_n} \in A_n(m) \mid f_{n,r}(a_0, \dots, a_n) = e_r \text{ for } 1 \leq r \leq k\}.$$

Each of these  $Q$ -posets carries several interesting structures. First, there is a natural rank-flipping involution  $\tau_n$ , which is defined on  $A_n$  as follows:

$$\tau_n(z_i) = z_{n-i}.$$

Furthermore, there is an explicit “raising and lowering” algorithm which provides a covering of  $Q(m, n, e_1, \dots, e_k)$  by (non-symmetric) saturated chains. If  $m > 2e_1 - e_2$ , then the chains in this covering are disjoint. There is also natural embedding

$$\epsilon : Q = Q(m, n, e_1, \dots, e_k) \rightarrow Q' = Q(2m - e_1, n + 2, m, e_1, \dots, e_k)$$

given by multiplication by  $z_{n+2}^{m-e_1}$ . Therefore, applying  $\epsilon$  and  $\tau_{n+2} \circ \epsilon$  to any symmetric chain in  $Q$  gives us a pair of chains which form the opposite sides of a “rectangle” in  $Q'$ . As we increase the degree, the length of this rectangle increases along with the number of translated copies of  $Q$  at the “top” and “bottom” the rectangle. It is straightforward to write down a symmetric chain decomposition for such a poset once the number of translated copies of  $Q$  in  $Q'$  is large enough compared to the length of the original chain in  $Q$ . Therefore, by induction we conclude that each  $Q(m, n, e_1, \dots, e_k)$  is a symmetric chain order if  $m$  is sufficiently large.

It follows that each  $L(m, n)$  has a generic “regular” part and a special “singular” part:

$$L(m, n) = L(m, n)^{reg} \oplus L(m, n)^{sing}$$

where  $L(m, n)^{reg}$  is a symmetric chain order. However, the intricacies of  $L(m, n)^{sing}$  are such that there is no obvious way to extend the symmetric chain decomposition of  $L(m, n)^{reg}$  to all of  $L(m, n)$ .

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## 2. SECANT IDEALS OF THE DEGENERATE RATIONAL NORMAL CURVE

Let  $A_n = \mathbb{C}[z_0, \dots, z_n]$  be the coordinate ring of  $\mathbb{P}^n$ . Let  $A_n(m)$  denote the set of monomials of degree  $m$  in  $A_n$ . Define a partial order on  $A_n(m)$  as follows:

$$z_0^{a_0} \dots z_n^{a_n} \leq z_0^{b_0} \dots z_n^{b_n} \iff \sum_{i=j}^n a_i \leq \sum_{i=j}^n b_i \text{ for } 1 \leq j \leq n.$$

This partial order is induced by the following action of  $\mathfrak{sl}_2\mathbb{C} = \mathbb{C}\langle H, E, F \rangle$  on the irreducible representation  $\mathbb{C}\langle z_0, \dots, z_n \rangle$ :

$$H(z_i) = n - 2i \quad E(z_i) = iz_{i-1} \quad F(z_i) = (n - i)z_{i+1}.$$

**2.1. Proposition.** There is a natural isomorphism of posets

$$L(m, n) \simeq A_n(m).$$

*Proof.* Define the map as follows:

$$\begin{aligned} L(m, n) &\rightarrow A_n(m) \\ (0 \leq \lambda_1 \leq \dots \leq \lambda_m \leq n) &\mapsto \prod_{i=1}^m z_{\lambda_i}. \end{aligned}$$

The inverse map is given by:

$$z_0^{a_0} \dots z_n^{a_n} \mapsto (\underbrace{0, \dots, 0}_{a_0 \text{ times}}, \underbrace{1, \dots, 1}_{a_1 \text{ times}}, \dots, \underbrace{n, \dots, n}_{a_n \text{ times}}).$$

The covering relations in  $L(m, n)$  are of the form:

$$(\lambda_0, \dots, \lambda_m) \leq (\lambda_0, \dots, \lambda_i + 1, \dots, \lambda_m)$$

while the covering relations in  $A_n(m)$  correspond to changing  $z_j$  to  $z_{j+1}$ :

$$z_0^{a_0} \dots z_n^{a_n} \leq z_0^{a_0} \dots z_j^{a_j-1} z_{j+1}^{a_{j+1}+1} \dots z_n^{a_n}$$

for some  $0 \leq j \leq n-1$ . □

Note that the highest weight elements in  $A_n(m)$  correspond to the lowest rank elements of  $L(m, n)$ . For this reason, from now on we will simply refer to the weights of elements instead of their ranks. In this formulation, the rank-symmetry condition turns into the more natural condition that the weights are symmetric under negation.

Recall that the ideal of the rational normal curve  $\mathcal{C}_n \subset \mathbb{P}^n$  is generated by the set of maximal minors of the  $(2 \times n)$ -Hankel matrix:

$$H_{n,1} = \begin{bmatrix} z_0 & z_1 & \dots & z_{n-1} \\ z_1 & z_2 & \dots & z_n \end{bmatrix}.$$

Furthermore, the ideal of the  $r$ -th secant variety of  $\mathcal{C}_n$  is generated by the set of maximal minors of the  $(r+1) \times (n-r+1)$  Hankel matrix:

$$H_{n,r} = \begin{bmatrix} z_0 & z_1 & \dots & z_r \\ z_1 & z_2 & \dots & z_{r+1} \\ \vdots & \vdots & & \vdots \\ z_{n-r} & z_{n-r+1} & \dots & z_n \end{bmatrix}$$

where  $r \leq k = \lfloor \frac{n}{2} \rfloor$ . Let  $I_{n,r}$  denote the initial ideal (with respect to any diagonal term order) of the ideal of the  $r$ -th secant variety of  $\mathcal{C}_n$ . The minimal generators of  $I_{n,r}$  correspond to initial monomials of Hankel determinants:

$$I_{n,r} = (\{z_{i_0} \dots z_{i_r} \mid i_j + 1 < i_{j+1} \text{ for } 0 \leq j \leq r-1\}).$$

For ease of notation, we define  $I_n = I_{n,1}$  and  $I_{n,0} = \mathfrak{m} = (z_0, \dots, z_n)$ . Note that  $I_{n,r}$  is a squarefree monomial ideal whose minimal generators all have degree  $r+1$ . Also, it turns out that  $I_{n,r}$  is equal to the  $r$ -th secant ideal of  $I_n$  [6](Section 6.1).

In order to work with the symbolic powers  $I_{n,r}^{(s)}$ , we need an explicit description of the (unique) irredundant irreducible decomposition of  $I_{n,r}$ . Our description will be in terms of the associated simplicial complex. Let  $\Delta_n$  denote the set of subsets of  $\{0, \dots, n\}$ . For any  $F \in \Delta_n$ , we define

$$z^F = \prod_{i \in F} z_i \quad \text{and} \quad \mathfrak{m}^F = \sum_{i \in F} (z_i).$$

Let  $\Gamma \subset \Delta_n$  be an abstract simplicial complex, so  $S \subset T \in \Gamma \implies S \in \Gamma$ . The Stanley-Reisner ideal of  $\Gamma$  is defined as [3]:

$$I_\Gamma = \langle z^F \mid F \notin \Gamma \rangle = \bigcap_{\overline{F} \in \Gamma} \mathfrak{m}^{\overline{F}}.$$

Let  $G(n, r)$  be the set of simplices corresponding to the minimal generators of  $I_{n,r}$ :

$$G(n, r) = \{\{i_0, \dots, i_r\} \in \Delta_n \mid i_j + 1 < i_{j+1} \text{ for } 0 \leq j \leq r-1\}.$$

The elements of  $G(n, r)$  are called 1-chains [1].

Note that  $I_{n,r}$  is the Stanley-Reisner ideal of the following simplicial complex:

$$\Gamma_{n,r} = \{F \mid S \not\subseteq F \text{ for any } S \in G_{n,r}\}$$

Note that  $\overline{F} \in \Gamma_{n,r} \iff F \cap S \neq \emptyset$  for all  $S \in G_{n,r}$ . In other words,

$$I_{n,r} = \bigcap_{\overline{F} \in \Gamma_{n,r}} \mathfrak{m}^{\overline{F}}$$

where the intersection is over vertex covers of the hypergraph  $G(n, r)$ . In order to make this decomposition irredundant, we need to find the minimal vertex covers of  $G(n, r)$ . Equivalently, we need to find the maximal simplices of  $\Gamma_{n,r}$ . We give such a description in terms of the simplicial complex  $\Gamma_n = \Gamma_{n,1}$ .

**2.2. Proposition.**  $\Gamma_n$  is the simplicial complex associated to the path graph:

$$\Gamma_n = \{\emptyset, \{0\}, \dots, \{n\}, \{0, 1\}, \{1, 2\}, \dots, \{n-1, n\}\}$$

*Proof.* Since all the generators of  $I_n$  are quadratic, clearly  $\emptyset, \{0\}, \dots, \{n\} \in \Gamma_n$ . Any edge of the form  $\{i, i+1\}$  is in  $\Gamma_n$  since it is not in  $G_{n,1}$ . On the other hand, any  $S \in \Delta_n$  with  $|S| \geq 3$  must contain some subset  $\{i, j\}$  such that  $i+1 < j$ , so  $S \notin \Gamma_n$ .  $\square$

**2.3. Remark.** By [5](Remark 2.9), the simplices of  $\Gamma_{n,r}$  are obtained by choosing  $r$  simplices from  $\Gamma_n$  and taking their union:

$$\Gamma_{n,r} = \{F_1 \cup \dots \cup F_r \mid F_i \in \Gamma_n\}.$$

In particular,  $|T| \leq 2r$  for any  $T \in \Gamma_{n,r}$ .

**2.4. Lemma.** For any  $S \in \Gamma_{n,r}$  there exist  $r$  disjoint edges  $T_1, \dots, T_r \in \Gamma_n$  such that  $S \subset T_1 \sqcup \dots \sqcup T_r$ . Consequently, each maximal simplex of  $\Gamma_{n,r}$  is equal to the complement of the union of  $r$  disjoint edges in  $\Gamma_n$ .

*Proof.* First, suppose  $S = F_1 \cup \dots \cup F_r \in \Gamma_{n,r}$  where some of the  $F_i$ 's are possibly empty. In this case, enlarge the simplex  $S$  by adding vertices until it is a union of  $r$  non-empty terms. Therefore, we may assume that  $S = F_1 \cup \dots \cup F_r \in \Gamma_{n,r}$  where each  $F_i$  is either a vertex or an edge of  $\Gamma_n$ . Then we can write

$$S = F'_1 \sqcup \dots \sqcup F'_r$$

where each  $F'_i$  is either a vertex or an edge of  $\Gamma_n$ . In fact, all possible non-disjoint unions of edges and vertices can be made disjoint without changing the number of terms:

$$\{a\} \cup \{a, b\} = \{a\} \sqcup \{b\} \quad \{a, b\} \cup \{b, c\} = \{a, b\} \sqcup \{c\} = \{a\} \sqcup \{b, c\}.$$

Therefore,  $S$  can be represented as a sequence of edges, vertices, and spaces. By the last equality above, we see that edges and vertices can “commute” without affecting the simplex  $S \in \Gamma_{n,r}$ . Moreover, we can get a simplex in  $\Gamma_{n,r}$  which contains  $S$  by changing a vertex and adjacent space into an edge. In algebraic terms, we are looking at the set of words in the alphabet  $\{e, v, s\}$  with the following substitution rules:

$$ev = ve \quad vs \mapsto e \quad sv \mapsto e.$$

Let  $n_x$  equal the number of times the letter  $x$  appears in a given word, for  $x \in \{e, v, s\}$ . Then we have the following restrictions for any  $S \in \Gamma_{n,r}$ :

$$n_e + n_v = r \quad 2n_e + n_v + n_s = n + 1.$$

Therefore,

$$2n_e = 2r - 2n_v = n + 1 - n_v - n_s \implies n_s = n + 1 - 2r + n_v$$

which implies that  $n_s \geq n_v$  since  $n + 1 - 2r \geq 0$ . So we can replace all possible occurrences of  $vs$  and  $sv$  with  $e$  (commuting  $v$  and  $e$  if necessary). The end result will be a sequence with only  $s$ 's and  $e$ 's where  $e$  appears  $n_e + n_v = r$  times. In other words, we have found  $r$  disjoint edges  $T_1, \dots, T_r \in \Gamma_n$  such that  $S \subset T_1 \sqcup \dots \sqcup T_r$ .  $\square$

**2.5. Corollary.** The irredundant irreducible decomposition of  $I_{n,r}$  is given by:

$$I_{n,r} = \bigcap_{0 \leq \lambda_0 \leq \dots \leq \lambda_{n-2r} \leq r} \mathfrak{m}^{\{2\lambda_0, 2\lambda_1+1, \dots, 2\lambda_{n-2r}+n-2r\}}.$$

*Proof.* The set of minimal vertex covers of  $G(n, r)$  is equal to

$$S(n, r) = \{\overline{T_1 \sqcup \dots \sqcup T_r} \mid T_1, \dots, T_r \text{ are disjoint edges of } \Gamma_n\}.$$

If we list the elements of  $\overline{T_1 \sqcup \dots \sqcup T_r}$  in increasing order, we see that each term will be sandwiched between sequences of consecutive edges chosen from  $\Gamma_n$ . Therefore,

$$S(n, r) = \{\{2\lambda_0, 2\lambda_1 + 1, \dots, 2\lambda_{n-2r} + n - 2r\} \mid 0 \leq \lambda_0 \leq \dots \leq \lambda_{n-2r} \leq r\}$$

and

$$I_{n,r} = \bigcap_{F \in S(n,r)} \mathfrak{m}^F = \bigcap_{0 \leq \lambda_0 \leq \dots \leq \lambda_{n-2r} \leq r} \mathfrak{m}^{\{2\lambda_0, 2\lambda_1+1, \dots, 2\lambda_{n-2r}+n-2r\}}.$$

□

**2.6. Corollary.** The tropical polynomials associated to  $I_{n,r}$ :

$$f_{n,r}(a_0, \dots, a_n) = \min_{0 \leq \lambda_0 \leq \dots \leq \lambda_{n-2r} \leq r} \sum_{j=0}^{n-2r} a_{2\lambda_j+j}$$

satisfy the following property:

$$z_0^{a_0} \dots z_n^{a_n} \in I_{n,r}^{(s)} \iff f_{n,r}(a_0, \dots, a_n) \geq s.$$

*Proof.* A squarefree monomial ideal  $I$  with the irredundant irreducible decomposition

$$I = \bigcap_F \mathfrak{m}^F$$

has the following symbolic powers:

$$I^{(s)} = \bigcap_F (\mathfrak{m}^F)^s.$$

Therefore,

$$z_0^{a_0} \dots z_n^{a_n} \in I^{(s)} \iff z_0^{a_0} \dots z_n^{a_n} \in (\mathfrak{m}^F)^s \text{ for all } F.$$

Now,

$$(\mathfrak{m}^F)^s = \left( \left\{ \prod_{i \in F} z_i^{b_i} \mid \sum_{i \in F} b_i = s \right\} \right)$$

and so:

$$z_0^{a_0} \dots z_n^{a_n} \in (\mathfrak{m}^F)^s \iff \sum_{i \in F} a_i \geq s.$$

Therefore,

$$z_0^{a_0} \dots z_n^{a_n} \in I_{n,r}^{(s)} \iff \sum_{j=0}^{2n-r} a_{2\lambda_j+j} \geq s \text{ for all } 0 \leq \lambda_0 \leq \dots \leq \lambda_{n-2r} \leq r.$$

□

**2.7. Example.** If  $n = 5$ , then  $I_{5,1} = (z_0 z_2, z_0 z_3, z_0 z_4, z_0 z_5, z_1 z_3, z_1 z_4, z_1 z_5, z_2 z_4, z_2 z_5, z_3 z_5)$  and  $I_{5,2} = (z_0 z_2 z_4, z_0 z_2 z_5, z_0 z_3 z_5, z_1 z_3 z_5)$  have irredundant irreducible decompositions:

$$I_{5,1} = (z_0, z_1, z_2, z_3) \cap (z_0, z_1, z_2, z_5) \cap (z_0, z_1, z_4, z_5) \cap (z_0, z_3, z_4, z_5) \cap (z_2, z_3, z_4, z_5)$$

$$I_{5,2} = (z_0, z_1) \cap (z_0, z_3) \cap (z_0, z_5) \cap (z_2, z_3) \cap (z_2, z_5) \cap (z_4, z_5)$$

and the corresponding tropical polynomials are:

$$f_{5,1} = \min(a_0 + a_1 + a_2 + a_3, a_0 + a_1 + a_2 + a_5, a_0 + a_1 + a_4 + a_5, a_0 + a_3 + a_4 + a_5, a_2 + a_3 + a_4 + a_5)$$

$$f_{5,2} = \min(a_0 + a_1, a_0 + a_3, a_0 + a_5, a_2 + a_3, a_2 + a_5, a_4 + a_5).$$

**2.8. Proposition.** If  $a, b \in \mathbb{Z}_{\geq 0}^{n+1}$ , then

$$f_{n,r}(a+b) \geq f_{n,r}(a) + f_{n,r}(b).$$

*Proof.* Let  $\mu_a = z_0^{a_0} \dots z_n^{a_n}$  and  $\mu_b = z_0^{b_0} \dots z_n^{b_n}$  be the monomials associated to the vectors  $a = (a_0, \dots, a_n)$  and  $b = (b_0, \dots, b_n)$ . Suppose  $f_{n,r}(a) = s$  and  $f_{n,r}(b) = t$ . Then  $\mu_a \in I_{n,r}^{(s)} - I_{n,r}^{(s+1)}$  and  $\mu_b \in I_{n,r}^{(t)} - I_{n,r}^{(t+1)}$ , and so:

$$\mu_a \mu_b \in I_{n,r}^{(s)} I_{n,r}^{(t)} \subset I_{n,r}^{(s+t)}.$$

Therefore,

$$f_{n,r}(a+b) \geq s+t = f_{n,r}(a) + f_{n,r}(b).$$

□

**2.9. Remark.** Note that  $G(n, 1)$  is equal to the set of 2-element antichains in the  $n$ -th zig-zag poset, so  $I_n$  a differentially perfect ideal [6] (Section 6.1). Therefore:

$$I_{n,r}^{(s)} = \sum_{\lambda_1 + \dots + \lambda_j = s} \prod_{i=1}^j I_{n,r+\lambda_i-1}$$

i.e. we can express the symbolic powers of any secant ideal of  $I_n$  as a sum of products of secant ideals of  $I_n$ .

### 3. TROPICAL DECOMPOSITION

Let  $m, n \geq 0$  and  $k = \lfloor \frac{n}{2} \rfloor$ . The level sets of the tropical polynomials  $\{f_{n,r}\}_{1 \leq r \leq k}$  give a canonical decomposition of  $A_n(m)$ :

$$Q(m, n, e_1, \dots, e_k) = \{z_0^{a_0} \dots z_n^{a_n} \in A_n(m) \mid f_{n,r}(a_0, \dots, a_n) = e_r \text{ for } 1 \leq r \leq k\}$$

The partial order on  $Q(m, n, e_1, \dots, e_k)$  is defined as the restriction of the partial order on  $A_n(m)$ .

Since  $\Gamma_n$  has an automorphism of order 2 which sends  $i$  to  $n-i$ , we see that  $I_{n,r}$  and  $f_{n,r}$  are invariant under the following automorphism:

$$\tau_n : A_n \rightarrow A_n$$

$$\tau_n(z_i) = z_{n-i}.$$

It follows that  $\tau_n$  restricts to a rank-flipping involution of  $Q(m, n, e_1, \dots, e_k)$ . In particular,  $Q(m, n, e_1, \dots, e_k)$  is a rank-symmetric, centered subposet of  $L(m, n)$ .

Note that among the minimal generators of  $I_{n,r}$ , there is a unique monomial of highest weight, namely:

$$\mu_{n,r} = z_0 z_2 z_4 \dots z_{2r} = \prod_{i=0}^r z_{2i}$$

If  $\mu = z_0^{a_0} \dots z_n^{a_n}$ , we will sometimes abuse notation and write  $f_{n,r}(\mu)$  instead of  $f_{n,r}(a_0, \dots, a_n)$ .

### 3.1. Proposition.

(1) If  $\mu = z_0^{a_0} \dots z_n^{a_n} \in Q(m, n, e_1, \dots, e_k)$ , then:

$$m - e_1 = \max_{0 \leq i \leq n-1} (a_i + a_{i+1}).$$

(2)  $f_{n,r}(\mu_{n,s}) = \max(s + 1 - r, 0)$  and:

$$\mu_{n,r} \in Q(r + 1, n, r, r - 1, \dots, 1, 0, \dots, 0).$$

(3) The functions  $f_{n,r}$  are additive on highest weight generators of  $I_{n,r}$ :

$$f_{n,r}\left(\prod_{j=0}^k \mu_{n,j}^{d_j}\right) = \sum_{j=r}^k (j + 1 - r)d_j.$$

*Proof.* (1) Note that  $e_0 = a_0 + \dots + a_n = m$ . By definition, we have:

$$e_1 = \min_{0 \leq i \leq n-1} (m - (a_i + a_{i+1})) = m - \max_{0 \leq i \leq n-1} (a_i + a_{i+1}).$$

Therefore:

$$m - e_1 = \max_{0 \leq i \leq n-1} (a_i + a_{i+1}).$$

In other words, we can calculate  $e_1$  by finding an edge of  $\Gamma_n$  which covers the two largest possible adjacent terms in  $(a_0, \dots, a_n)$ .

(2) Let

$$g_{n,s} = (\underbrace{1, 0, \dots, 1, 0}_{\text{length } 2s}, 1, \underbrace{0, \dots, 0}_{\text{length } n-2s}) \in \mathbb{Z}_{\geq 0}^{n+1}$$

be the lattice vector corresponding to the monomial  $\mu_{n,s}$ . By definition,

$$f_{n,r}(g_{n,s}) = \min_{0 \leq \lambda_0 \leq \dots \leq \lambda_{n-2r} \leq r} \sum_{j=0}^{n-2r} g_{n,s}(2\lambda_j + j).$$

In other words, for each choice of  $r$  disjoint edges in  $\Gamma_n$ , we sum over the terms in  $g_{n,s}$  that are not covered by the edges, and look for the smallest possible outcome. In other words, we should try to cover as many non-zero terms of  $g_{n,s}$  as possible. Since all the non-zero terms of  $g_{n,s}$  are two steps away from each other, each edge can only cover one such vertex. Therefore, our sum will count the number of vertices in the support of  $g_{n,s}$  which cannot be covered by  $r$  edges:

$$f_{n,r}(g_{n,s}) = \max(s + 1 - r, 0).$$

Since  $\deg(\mu_{n,r}) = r + 1$ , and  $f_{n,j}(\mu_{n,r}) = \max(r + 1 - j, 0)$ , we see that:

$$\mu_{n,r} \in Q(r + 1, n, r, r - 1, \dots, 1, 0, \dots, 0).$$

(3) Let

$$\mu = \prod_{j=0}^k \mu_{n,j}^{d_j} \quad \text{and} \quad v = (v_0, \dots, v_n) = \sum_{j=0}^k d_j g_{n,j}.$$



Note that

$$\mu = \prod_{j=0}^k \prod_{i=0}^j z_{2i}^{d_j} = \prod_{i=0}^k \prod_{j=i}^k z_{2i}^{d_j} = \prod_{i=0}^k z_{2i}^{v_{2i}}$$

where

$$v_{2i} = \sum_{j=i}^k d_j.$$

We calculate the value of  $f_{n,r}(v)$  by covering the largest terms in  $v$  using  $r$  disjoint edges in  $\Gamma_n$ . As before, each edge in  $\Gamma_n$  can only cover one vertex at a time, so we will end up summing over those  $v_{2s}$  such that  $2s > 2r$ :

$$f_{n,r}(v) = \sum_{i=r}^k \sum_{j=i}^k d_j = \sum_{j=r}^k \sum_{i=r}^j d_j = \sum_{j=r}^k (j+1-r)d_j.$$

□

The following proposition is useful for translating between the values of the tropical polynomials and factorizations of the corresponding monomials (see Remark 3.3):

**3.2. Proposition.** Let  $d_0, \dots, d_k \in \mathbb{Z}_{\geq 0}$ . Then the following are equivalent:

(1) For all  $0 \leq r \leq k$ ,

$$d_r = e_r - 2e_{r+1} + e_{r+2}$$

where  $e_j = 0$  if  $j > k$ .

(2) For all  $0 \leq r \leq k$ ,

$$e_r = \sum_{j=r}^k (j+1-r)d_j.$$

*Proof.* (1)  $\implies$  (2) :

$$\begin{aligned} \sum_{j=r}^k (j+1-r)d_j &= \sum_{j=r}^k (j+1-r)(e_j - 2e_{j+1} + e_{j+2}) \\ &= \sum_{j=r}^k (j+1-r)e_j - 2 \sum_{j=r}^{k-1} (j+1-r)e_{j+1} + \sum_{j=r}^{k-2} (j+1-r)e_{j+2} \\ &= \sum_{j=r}^k (j+1-r)e_j - 2 \sum_{j=r+1}^k (j-r)e_j + \sum_{j=r+2}^k (j-1-r)e_j \\ &= e_r + \sum_{j=r+1}^k (j+1-r-2j+2r+j-1-r)e_j \\ &= e_r \end{aligned}$$

(2)  $\implies$  (1) :

$$\begin{aligned}
e_r - 2e_{r+1} + e_{r+2} &= \sum_{j=r}^k (j+1-r)d_j - \sum_{j=r+1}^k 2(j+1-r)d_j + \sum_{j=r+2}^k (j+1-r)d_j \\
&= d_r + 2d_{r+1} - 2d_{r+1} + \sum_{j=r+2}^k ((j+1-r) - 2(j+1-r) + (j+1-r))d_j \\
&= d_r
\end{aligned}$$

□

**3.3. Remark.** It is worth mentioning that our tropical decomposition is equivalent to A. Conca's "canonical decomposition" [1], which gives a unique factorization of each monomial in  $A_n$  as a product of minimal generators of  $I_{n,r}$ , for  $0 \leq r \leq k$ . Let us recall the construction. First, we order the minimal generators of each  $I_{n,r}$  lexicographically by  $z_0 > z_1 > \dots > z_n$ , and then by degree, so each generator of  $I_{n,r}$  is larger than each generator of  $I_{n,r-1}$ . Given a monomial  $\mu$ , we factor out the largest generator of  $\{I_{n,r}\}_{0 \leq r \leq k}$  which divides  $\mu$ . Then we repeat this process for the monomial  $\mu/\mu_1$ , and so on. In the end, we get the canonical decomposition:

$$\mu = \mu_1 \dots \mu_t$$

where  $\mu_i$  is a minimal generator of  $I_{n,r_i}$  for  $1 \leq i \leq t$ . Note that  $r_1 \geq \dots \geq r_t$ , and that  $\deg(\mu_i) = r_i + 1$ . We can represent this factorization of  $\mu$  as a tableau whose  $i$ -th row is the 1-chain corresponding to  $\mu_i$ . Such a tableau comes from a canonical decomposition if and only if each row is a 1-chain with entries in  $\{0, \dots, n\}$  and, for any entry  $a$ , either  $a$  or  $a-1$  appears in each preceding row.

Let  $M(m, n, d_1, \dots, d_k)$  denote the set of monomials of degree  $m$  in  $A_n$  whose canonical decomposition contains  $d_r$  minimal generators of  $I_{n,r}$ . We claim that:

$$M(m, n, d_1, \dots, d_k) = Q(m, n, e_1, \dots, e_k)$$

where:

$$e_r = \sum_{j=r}^k (j+1-r)d_j.$$

It suffices to prove that if  $\mu = \mu_1 \dots \mu_t$  is the canonical decomposition of  $\mu$ , then:

$$f_{n,r}(\mu) = \sum_{i=1}^t f_{n,r}(\mu_i).$$

To see this, note that for any entry  $a$  in the last row of the corresponding tableau, either  $a$  or  $a-1$  must appear in the preceding row. We need to choose  $r$  disjoint edges from  $\Gamma_n$  covering the largest possible entries in the tableau. An edge can cover both  $a$  and  $a-1$ , so we should begin by covering the entries in the last row. Also, we can only cover one entry in each row since the rows are 1-chains, i.e. the difference between non-zero entries in each row is at least two. In this way, we see that each edge

choice will cover exactly one entry in each preceding row, which implies the additivity property above.

Now define the *order of vanishing* of a monomial  $\mu \in A_n$  on a monomial ideal  $I \subset A_n$  to be:

$$\text{ord}_I(\mu) = \max\{i \geq 0 \mid \mu \in I^i\}.$$

The order of vanishing of  $\mu$  on  $I_{n,r}$  can be calculated in terms of the number times each minimal generator appears in the canonical decomposition of  $\mu$ . Indeed, if the canonical decomposition of  $\mu$  contains  $d_r$  minimal generators of  $I_{n,r}$ , then:

$$\text{ord}_{I_{n,r}}(\mu) = \sum_{j=0}^k d_j \left\lfloor \frac{j+1}{r+1} \right\rfloor.$$

**3.4. Corollary.** We can express the order of vanishing on  $I_{n,r}$  in terms of tropical polynomials:

$$\text{ord}_{I_{n,r}} = \sum_{j=0}^k (f_{n,j} - 2f_{n,j+1} + f_{n,j+2}) \left\lfloor \frac{j+1}{r+1} \right\rfloor.$$

**3.5. Remark.** It follows from Propositions 3.1 and 3.2 that the monomial:

$$\mu(m, n, e_1, \dots, e_k) = \prod_{j=0}^k \mu_{n,j}^{d_j}$$

lies in  $Q(m, n, e_1, \dots, e_k)$ , where  $d_j = e_j - 2e_{j+1} + e_{j+2}$  and  $e_j = 0$  if  $j > k$ . We also define the monomial:

$$\nu(m, n, e_1, \dots, e_k) = \tau_n(\mu(m, n, e_1, \dots, e_k)) = \prod_{j=0}^k \tau_n(\mu_{n,j})^{d_j} = \prod_{j=0}^k \prod_{i=0}^j z_{n-2i}^{d_j}.$$

**3.6. Lemma.** For any  $d \geq 0$ , multiplication by  $z_{n+2}^{m-e_1+d}$  induces an embedding of posets

$$\epsilon_n^d : Q(m, n, e_1, \dots, e_k) \rightarrow Q(2m - e_1 + d, n + 2, m, e_1, \dots, e_k)$$

which sends  $\nu(m, n, e_1, \dots, e_k)$  to  $z_{n+2}^d \nu(2m - e_1, n + 2, m, e_1, \dots, e_k)$ .

*Proof.* Note that:

$$z_{n+2} \tau_n(\mu_{n,j}) = z_{n+2} \prod_{i=0}^j z_{n-2i} = \prod_{i=0}^{j+1} z_{n+2-2i} = \tau_{n+2}(\mu_{n+2,j+1}).$$

Therefore,

$$\nu(m, n, e_1, \dots, e_k) z_{n+2}^{m-e_1} = \prod_{j=0}^k z_{n+2}^{d_j} \prod_{i=0}^j z_{n-2i}^{d_j} = \prod_{j=0}^k \tau_{n+2}(\mu_{n+2,j+1}^{d_j}).$$

Now look at the monomial

$$\mu' = \prod_{j=0}^{k+1} \mu_{n+2,j}^{d'_j}$$

where  $d'_j = d_{j-1}$  and  $d_{-1} = 0$ . For  $r > 0$ , we have:

$$e'_r = \sum_{j=r}^{k+1} (j+1-r)d'_j = \sum_{j=r}^{k+1} (j+1-r)d_{j-1} = \sum_{j=r-1}^k (j+1-(r-1))d_j = e_{r-1}$$

and

$$e'_0 = \sum_{j=0}^{k+1} (j+1)d_{j-1} = \sum_{j=0}^k (j+2)d_j = 2m - e_1$$

This shows that

$$\mu' \in Q(2m - e_1, n+2, m, e_1, \dots, e_k)$$

and so

$$\nu(m, n, e_1, \dots, e_k) z_{n+2}^{m-e_1} = \nu(2m - e_1, n+2, m, e_1, \dots, e_k).$$

Multiplying by  $z_{n+2}^d$ , we get

$$\nu(m, n, e_1, \dots, e_k) z_{n+2}^{m-e_1+d} = z_{n+2}^d \nu(2m - e_1, n+2, m, e_1, \dots, e_k).$$

Next we show that multiplication by  $z_{n+2}^{m-e_1+d}$  defines a map

$$\epsilon_n^d : Q(m, n, e_1, \dots, e_k) \rightarrow Q(2m - e_1 + d, n+2, m, e_1, \dots, e_k).$$

Suppose  $(a_0, \dots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1}$  such that  $f_{n,r}(a_0, \dots, a_n) = e_r$  for  $0 \leq r \leq k$ . Recall that  $m - e_1$  is the largest part of the lattice vector which can be covered by a single edge in  $\Gamma_n$ :

$$m - e_1 = \max_{0 \leq i \leq n-1} (a_i + a_{i+1}).$$

Now consider the vector  $(a_0, \dots, a_n, 0, m - e_1 + d) \in \mathbb{Z}_{\geq 0}^{n+3}$ . Then:

$$f_{n+2,r+1}(a_0, \dots, a_n, 0, m - e_1 + d) = e_r$$

because covering the vertices  $n+1$  and  $n+2$  will remove the largest possible term from the sum due to the inequalities  $a_i + a_{i+1} \geq a_j + a_{j+1}$  for all  $0 \leq j \leq n-1$ .

Now multiplication by a fixed monomial is an injective map, and since the partial order is defined the same way in  $\mathbb{Z}_{\geq 0}^{n+1}$  and  $\mathbb{Z}_{\geq 0}^{n+3}$ , the map is actually an embedding of posets.  $\square$

#### 4. THE RAISING AND LOWERING ALGORITHM

The edges of the Hasse diagram of  $A_n(m)$  are colored by  $\{1, \dots, n\}$ , where the  $i$ -th color corresponds to the following covering relation:

$$(a_0, \dots, a_n) \rightarrow (a_0, \dots, a_{i-1} - 1, a_i + 1, \dots, a_n).$$

Let  $C$  be a saturated chain in  $A_n(m)$  and let  $x_0$  be the highest weight element in  $C$ . Then  $C$  can be represented by a sequence of colors  $(c_1, \dots, c_t)$  by starting with  $x_0$  and

reading downward. We say that a saturated chain  $C$  is *monotonic* if  $c_1 \leq \dots \leq c_t$ . Next we prove the fundamental covering property for tropical decomposition.

**4.1. Lemma.** There is a covering of  $Q(m, n, e_1, \dots, e_k)$  by monotonic saturated chains which start in  $\tau_n \epsilon_{n-2}^{d_0} Q(e_1, n-2, e_2, \dots, e_k)$  and end in  $\epsilon_{n-2}^{d_0} Q(e_1, n-2, e_2, \dots, e_k)$ . Moreover, if  $m > 2e_1 - e_2$ , then these chains are pairwise disjoint.

*Proof.* First, we describe a way to move downward in the poset  $Q(m, n, e_1, \dots, e_k)$  by monotonic saturated chains. Let  $(a_0, \dots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1}$  such that  $f_{n,r}(a_0, \dots, a_n) = e_r$  for  $0 \leq r \leq k$ . Recall that:

$$m - e_1 = \max_{0 \leq i \leq n-1} (a_i + a_{i+1}).$$

Let  $0 \leq i \leq n-1$  such that  $a_i + a_{i+1} = m - e_1$ . Then  $a_{i+1} + a_{i+2} \leq a_i + a_{i+1}$ , so  $a_{i+2} \leq a_i$ . There are two cases:

- (i) If  $a_i = a_{i+2}$ , then we can move the edge one step to the right and cover  $\{a_{i+1}, a_{i+2}\}$  instead.
- (ii) If  $a_i > a_{i+2}$ , then we can move downward by the color  $i+1$ , which yields the new vector  $(\dots, a_i - 1, a_{i+1} + 1, a_{i+2}, \dots)$ . After we do this move  $(a_i - a_{i+2})$  times we get the following vector:

$$(\dots, a_{i+2}, a_{i+1} + a_i - a_{i+2}, a_{i+2}, \dots)$$

and then we are back in case (i).

By repeated application of (i) and (ii), we construct a monotonic saturated chain in  $Q(m, n, e_1, \dots, e_k)$  whose maximal element is  $z_0^{a_0} \dots z_n^{a_n}$ . This process terminates when  $a_{n-1} = 0$  and  $a_n = m - e_1$ , which means we have reached an element of  $\epsilon_{n-2}^{d_0} Q(e_1, n-2, e_2, \dots, e_k)$ . Similarly, we have a process for moving upward in  $Q(m, n, e_1, \dots, e_k)$  until we reach an element of  $\tau_n \epsilon_{n-2}^{d_0} Q(e_1, n-2, e_2, \dots, e_k)$ . Therefore, at each element of  $\tau_n \epsilon_{n-2}^{d_0} Q(e_1, n-2, e_2, \dots, e_k)$ , we begin a monotonic saturated chain which ends at an element of  $\epsilon_{n-2}^{d_0} Q(e_1, n-2, e_2, \dots, e_k)$ , and these “transversal” chains form a covering of  $Q(m, n, e_1, \dots, e_k)$ .

In particular, it follows that  $\mu(m, n, e_1, \dots, e_k)$  (resp.  $\nu(m, n, e_1, \dots, e_k)$ ) is the unique monomial in  $Q(m, n, e_1, \dots, e_k)$  of highest (resp. lowest) weight.

We now give an alternate description of these chains. Let  $\mu \in \tau_n \epsilon_{n-2}^{d_0} Q(e_1, n-2, e_2, \dots, e_k)$ . Let  $C_\mu$  be the chain in  $Q(m, n, e_1, \dots, e_k)$  constructed above that starts at  $\mu$ . Let  $c$  be the first color in the color sequence for  $C_\mu$ . The lattice vector corresponding to  $\mu$  looks like:

$$(a_0, a_1, \dots, a_{c-3}, a_{c-2}, a_{c-1}, a_c, \dots)$$

and by construction  $c$  is the first spot where  $a_{c-1} > a_{c+1}$ . Therefore,  $a_i + a_{i+1} = m - e_1$  for all  $0 \leq i \leq c-1$ , otherwise we could still move upward in the chain. Now if we apply the color  $c$ , we get the vector:

$$(a_0, a_1, \dots, a_{c-3}, a_{c-2}, a_{c-1} - 1, a_c + 1, \dots).$$

which will be the starting point for a chain whose first color is  $c - 2$ . In other words, we can generate new transversal chains from old by subtracting 2 from the smallest color possible while preserving monotonicity. As a consequence, the transversal chain starting at  $\mu(m, n, e_1, \dots, e_k)$  will end at  $\epsilon_{n-2}^{d_0} \mu(e_1, n - 2, e_2, \dots, e_k)$ . If  $(c_1, \dots, c_t)$  is a color sequence from  $\mu(m, n, e_1, \dots, e_k)$  to  $\tau_n \epsilon_{n-2}^{d_0} \mu$  for some  $\mu \in Q(e_1, n - 2, e_2, \dots, e_k)$ , then the transversal chain starting at  $\tau_n \epsilon_{n-2}^{d_0} \mu$  will end at the point of  $\epsilon_{n-2}^{d_0} Q(e_1, n - 2, e_2, \dots, e_k)$  given by applying the color sequence  $(c_1 - 2, \dots, c_t - 2)$  to  $\epsilon_{n-2}^{d_0} \mu(e_1, n - 2, e_2, \dots, e_k)$ . In other words, the transversal chain starting at  $\tau_n \epsilon_{n-2}^{d_0} \mu$  will end at  $\epsilon_{n-2}^{d_0} \tau_{n-2} \mu$ .

If  $m > 2e_1 - e_2$  then  $d_0 > 0$ . We show that the chains constructed above are pairwise disjoint. Indeed, the elements of  $\tau_n \epsilon_{n-2}^{d_0} Q(e_1, n - 2, e_2, \dots, e_k)$  are of the form  $z_0^{d_0} \mu$ , where  $\mu \in \tau_n \epsilon_{n-2}^0 Q(e_1, n - 2, e_2, \dots, e_k)$ . Therefore, from  $z_0^{d_0} \mu$  we must follow the color sequence  $(1^{d_0}, 2^{d_0}, \dots, n^{d_0})$  downward, working in tandem with the chain  $C_\mu$  whose starting point is  $\mu$ . The resulting color sequence will be  $(1^{d_0}, 2^{d_0}, \dots, n^{d_0})$  with the colors from  $C_\mu$  inserted in order to preserve monotonicity.

Let  $C$  be a monotonic saturated chain in  $Q(m, n, e_1, \dots, e_k)$  as constructed above, with color sequence  $(c_1, \dots, c_t)$  and highest weight element  $x_0$ . If  $c \geq 3$  and  $c = c_i$  for some  $1 \leq i \leq t$ , then we have a monotonic saturated chain  $C'$  whose monotonic color sequence is  $(c_1, \dots, c - 2, \dots, c_t)$  and whose highest weight element  $x'_0$  is covered by  $x_0$  by an edge with color  $c$ . It suffices to show that  $C$  and  $C'$  are disjoint.

Note that the color sequences of  $C$  and  $C'$  have the same underlying multiset of colors except that one contains  $c$  and the other contains  $c - 2$ . Now if  $d_0 > 0$ , then the terms of the color sequences of both  $C$  and  $C'$  can increase by at most one at each step, since they both insert colors into the base sequence  $(1^{d_0}, 2^{d_0}, \dots, n^{d_0})$  while preserving monotonicity. Therefore, we have two saturated chains starting at  $x_0$  whose color sequences are of the form  $(c_1, c_2, \dots, c_t, c - 2)$  and  $(c, c_1, c_2, \dots, c - 2, \dots, c_t)$ , and the two sequences share the same underlying multiset of colors. These chains will intersect after  $j$  steps if and only if their color sequences share the same multiset of colors in the first  $j$  terms. Since  $c \geq 3$  and  $c_1 = 1$ , this can only happen once we reach the color  $c$  in the first sequence. However, we have changed this color to  $c - 2$  in the second sequence, and the only way we can gain an extra  $c - 2$  in the first sequence is if  $j = t$ . Therefore,  $C$  and  $C'$  are disjoint.

□

**4.2. Theorem.**  $Q(m, n, e_1, \dots, e_k)$  is a symmetric chain order if  $m$  is sufficiently large.

*Proof.* We proceed by induction on  $n$ . If  $n = 0$  then  $Q(m, 0) \simeq L(m, 0)$  has only one element. If  $n = 1$ , then  $Q(m, 1) \simeq L(m, 1)$  is a single chain with  $m + 1$  vertices.

Suppose  $Q(m, n, e_1, \dots, e_k)$  is a symmetric chain order, and let  $\ell$  be the number of elements in its longest chain. We will construct a symmetric chain decomposition of  $Q(2m - e_1 + d_0, n + 2, e_1, \dots, e_k)$  for all  $d_0$  such that  $2d_0 \geq \ell - 1$ .

Indeed, if for any  $d_0 > 0$  we have a decomposition of  $Q(2m - e_1 + d_0, n + 2, e_1, \dots, e_k)$  into (non-symmetric) monotonic saturated chains which start in  $\tau_{n+2}\epsilon_n^{d_0}Q(m, n, e_1, \dots, e_k)$  and end in  $\epsilon_n^{d_0}Q(m, n, e_1, \dots, e_k)$ .

Let  $C_1, \dots, C_t$  be the chains in a symmetric chain decomposition of  $Q(m, n, e_1, \dots, e_k)$ . Note that  $\tau_n C_1, \dots, \tau_n C_t$  are the chains of the “dual” symmetric chain decomposition. Now:

$$Q(2m - e_1 + d_0, n + 2, e_1, \dots, e_k) \simeq \bigsqcup_{i=1}^t R_i$$

where  $R_i$  is a “rectangular” poset containing the “top” chain  $\tau_{n+2}\epsilon_n^{d_0}C_i$ , the “bottom” chain  $\epsilon_n^{d_0}\tau_n C_i$ , and all the “transversal” monotonic saturated chains between them. Furthermore,  $R_i$  contains  $d_0 + 1$  copies of  $\tau_{n+2}\epsilon_n^{d_0}C_i$  generated by application of the color “1” to each element of  $\tau_{n+2}\epsilon_n^{d_0}C_i$ , i.e. changing  $z_0$  to  $z_1$ . Similarly,  $R_i$  contains  $d_0 + 1$  copies of  $\epsilon_n^{d_0}\tau_n C_i$  generated by reverse application of the color “ $n + 2$ ” to each element of  $\epsilon_n^{d_0}C_i$ , i.e. changing  $z_{n+2}$  to  $z_{n+1}$ .

We claim that each  $R_i$  has a symmetric chain decomposition if  $2d_0 > \ell - 1$ . Indeed, we can follow the top chain  $\tau_{n+2}\epsilon_n^{d_0}C_i$  and then the transversal chain all the way to the minimal element, or follow the transversal chain from the maximal element and then follow the bottom chain  $\epsilon_n^{d_0}\tau_n C_i$ . So we have a pair of disjoint symmetric chains as the outer shell of  $R_i$ . If we remove this shell, we are left with a smaller rectangle, where we have removed the first and last edges from the top chain, the bottom chain, and each transversal chain. Note that we may no longer have a top and bottom chain in what remains because we cannot control what happens in the interior of  $R_i$ . However, since  $\tau_{n+2}\epsilon_n^{d_0}C_i$  and  $\epsilon_n^{d_0}\tau_n C_i$  have at most  $\ell$  elements, we only need to do this procedure at most  $\frac{\ell+1}{2}$  times. Now there are  $d_0 + 1$  copies of the top chain and the bottom chain in  $R_i$ , so if  $d_0 + 1 \geq \frac{\ell+1}{2}$ , then we are guaranteed to have a sufficient supply of top and bottom chains so that each shell of  $R_i$  is a disjoint union of a pair of symmetric chains.  $\square$

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